Fall 2007

1

Let $x =$ the volume of the can (in ounces)

Then $4\frac{1}{3}x + x = 64$

$5\frac{1}{3}x = 64$ Combine like terms

$\frac{16}{3}x = 64$ Convert mixed number to improper fraction

$x = 12$ multiply both sides by $\frac{3}{16}$

So the answer is **C. 12**

2

$(3\Delta 2)\Delta(2\Delta 3)$

$((3 * 2) + 2)\Delta((2 * 3) + 3)$ Definition of Δ

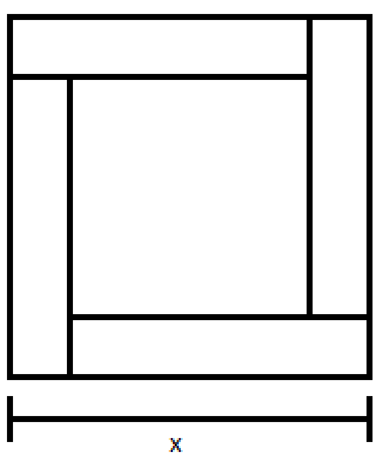
$8\Delta 9$

$(8 * 9) + 9$ Definition of Δ

81

So the answer is **D. 81**

3



If we define x as the length of a side of the large square, then the area of the large square is x^2. The area of the small square is $\frac{4}{9}$ of that, so it is $\frac{4}{9}x^2$. Thus a side of the small square is $\sqrt{\frac{4}{9}x^2} = \frac{2}{3}x$.

The length of a side of one of the rectangles is the average of the length of a side of the large square and a side of the small square, so is $\frac{\frac{2}{3}x+x}{2} = \frac{\frac{5}{3}x}{2} = \frac{5}{6}x$.

The ratio of a large side of one of the rectangles to a side of the small square is

$$\frac{\frac{5}{6}x}{\frac{2}{3}x} = \frac{\frac{5}{6}}{\frac{2}{3}} = \frac{5}{6} * \frac{3}{2} = \frac{15}{12} = \frac{5}{4}.$$

So the answer is **A. $\frac{5}{4}$**

4

$$\frac{1\text{ time}}{10\text{ minutes}} * \frac{60\text{ minutes}}{1\text{ hour}} * \frac{24\text{ hours}}{1\text{ day}} * \frac{7\text{ days}}{1\text{ week}} = \frac{1008\text{ times}}{1\text{ week}}$$

$1008 - 1000 = 8$

So the answer is **A. 8**

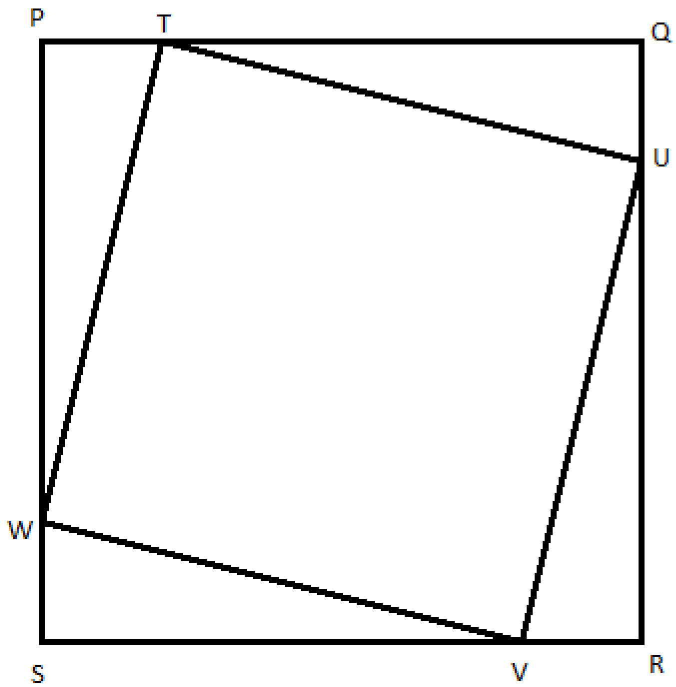
5

Since she has only nickels and dimes, the amount of money she has must be a multiple of \$0.05. Since replacing a dime for a nickel decreases the value, she will have the least dimes when she has the least money. So for her to have the least number of dimes possible she has to have \$1.75. Now we can set up our two equations and solve to find out how many dimes she has.

$N + D = 24$	She has two dozen coins
$.05N + .10D = 1.75$	The value is \$1.75
$5N + 10D = 175$	multiply by 100 to cancel decimals
$-5N - 5D = -120$	multiply first equation by -5 to set up addition
$5D = 55$	add above two equations, N cancels
$D = 11$	divide by 5

So the answer is **B. 11**

6



$PQ = QR = RS = SP = 10$

$PT = QU = RV = SW = 2$

Thus $TQ = UR = VS = WP = 8$

$TU = UV = VW = WT = \sqrt{8^2 + 2^2}$ by Pythagorean Theorem.

$= \sqrt{64 + 4} = \sqrt{68}$

Since all sides are equal, quadrilateral TUVW is a square. The area equals the length of a side squared, $\left(\sqrt{68}\right)^2$, which is 68

So the answer is **E. 68**

7

We are given the fact that s feet per minute $=\ s - 16$ inches per second. First we want to convert everything to similar units. Converting s feet per minute into inches per second:

$$s\,\frac{\text{feet}}{\text{minute}} * \frac{1\text{ minute}}{60\text{ seconds}} * \frac{12\text{ inches}}{1\text{ foot}} = \frac{12}{60}s\,\frac{\text{in}}{\text{sec}} = \frac{1}{5}s\,\frac{\text{in}}{\text{sec}}$$

Now, in inches per second, $\frac{1}{5}s = s - 16$

$-\frac{4}{5}s = -16$ subtract s from both sides

$-4s = -80$ multiply both sides by 5

$s = 20$ divide by -4

So the answer is **D. 20**

8

The formula for average is $\frac{\text{sum of terms}}{\text{\# of terms}}$, so we can convert our averages into equations.

$\frac{A+2B}{2} = 7$

$\frac{A+2C}{2} = 8$

$\frac{A+2B+A+2C}{2} = 15$ add 2 equations above

$\frac{2A+2B+2C}{2} = 15$ combine like terms

$A + B + C = 15$ reduce

$\frac{A+B+C}{3} = 5$ divide by 3 to get the average

So the answer is **C. 5**

9

Computing 6 digit perfect squares by hand is a long process, so for this problem we can let the calculator do the work for us. I assume you are using a TI 83 or similar, as those are the calculators I am most familiar with, and those are the calculators we borrow out at this campus (Tarrant County College Southeast).

First press y=, then enter X^2. Now just going to table to look at values will take a while because we want large 6 digit numbers, so press 2nd, and table set (TBL SET, above window), set the table to start at 1000 (1000^2 is 1,000,000, so our number will be less than that). Then press 2nd, table (graph), and press up until we see a number which has 1st and 3rd digits the same, all others unique. The largest such number we should see is 898704, which is 948^2.

$8 + 9 + 8 + 7 + 0 + 4 = 36$

So the answer is **E. 36**

10

Opening the door makes the shape of the door and door frame half of a rectangular prism. The longest distance from a point on the door frame to the bottom corner of the door will be at the top corner of the frame opposite the door, which is equivalent to the distance of the diagonal of the rectangular prism shape. This length is
$\sqrt{7^2 + 4^2 + 4^2} = \sqrt{49 + 16 + 16} = \sqrt{81} = 9$

So the answer is **B. 9**

11

$F = .4T$	Females are 40% of the total
$F + 3 = .44T$	replacement adds 3 females, leaves total the same
$3 = .04T$	second equation + -1 times the first
$75 = T$	multiply both sides by 25
$M = .6T$	
$M - F = x$	we are asked for the difference between the number of males and females
$.6T - .4T = x$	substitution
$.2T = x$	
$.2(75) = x$	substitution for T
$15 = x$	

So the answer is **C. 15**

12

The 2 saxophone parts have $2 * 1 = 2$ ways they can be assigned, and the 3 trumpet and 3 trombone parts each have $3 * 2 * 1 = 6$ ways they can be assigned. Total, there are $2 * 6 * 6 = 72$ different combinations.

So the answer is **B. 72**

13

$(2N)^2 + N = M$

$4N^2 + N = M$

$N(4N + 1) = M$

M can only be prime if N or 4N+1 = $\pm$1. Test all four cases.

IF $N = 1$

$4N + 1 = 5$

$N(4N + 1) = 1 * 5 = 5$, which is prime.

IF $N = -1$

$4N + 1 = -3$

$N(4N + 1) = (-1) * (-3) = 3$, which is prime

IF $4N + 1 = 1$

$4N = 0$

$N = 0$

$N(4N + 1) = 0 * 1 = 0$, which is not prime

IF $4N + 1 = -1$

$4N = -2$

$N = -\frac{1}{2}$, which is not an integer

So M is only prime if N=1 or N=-1, which is in 2 cases.

So the answer is **C. 2**

14

We can use the process of elimination to figure out who is partnered with who, then look for the largest difference. Starting out we have 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 and 16.

We can see that 16 must go with 9 (it cannot reach 36, or stay at 16 when added), and 8 must go with 1 (it cannot go with itself for 16, and cannot reach 25).

This gives us 8+1, 16+9, and we have 2 3 4 5 6 7 10 11 12 13 14 and 15 left.

Now that 9 is gone, 7 must go with 2, and with 1 gone, 15 must go with 10.

This gives us 8+1, 16+9, 7+2, and 15+10, and we are left with 3 4 5 6 11 12 13 and 14.

With 2 gone, 14 goes with 11, and with 10 gone, 6 goes with 3, giving us 6+3 and 14+11 and leaving 4 5 12 and 13.

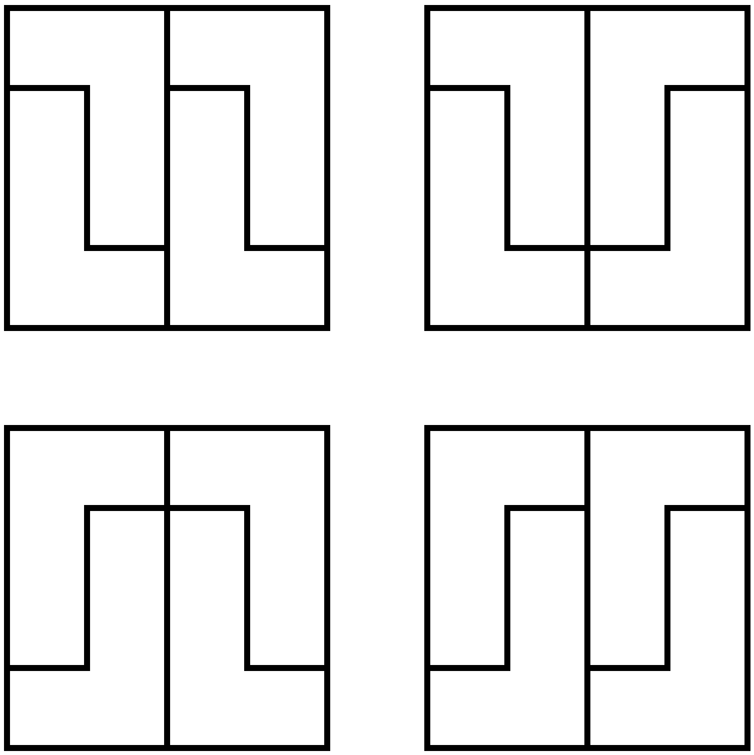
13 has to go with 12, leaving 4 with 5. So our final list is

1+8	9+16
2+7	10+15
3+6	11+14
4+5	12+13

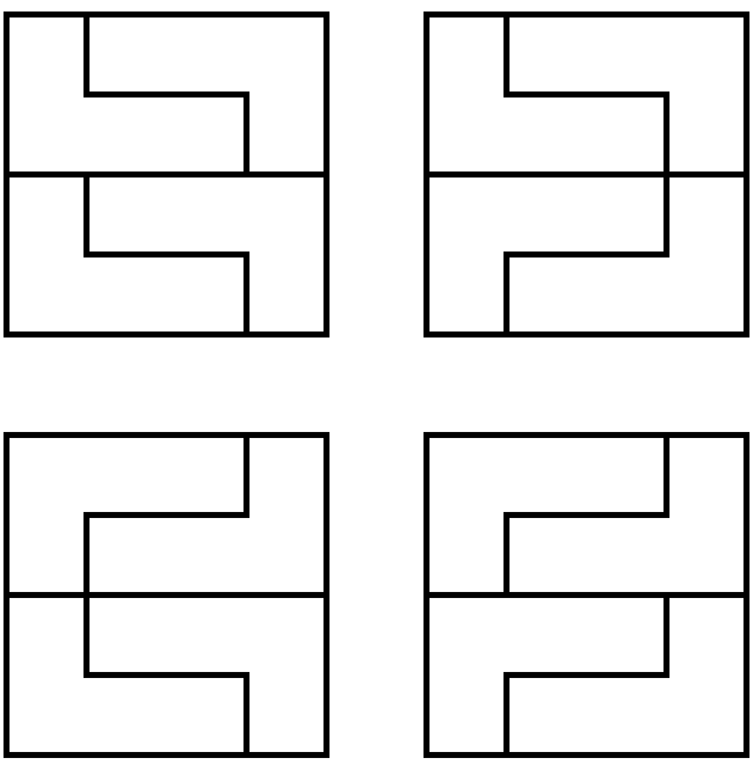
And the largest difference is 7, from 16-9 and 8-1.

So the answer is **B. 7**

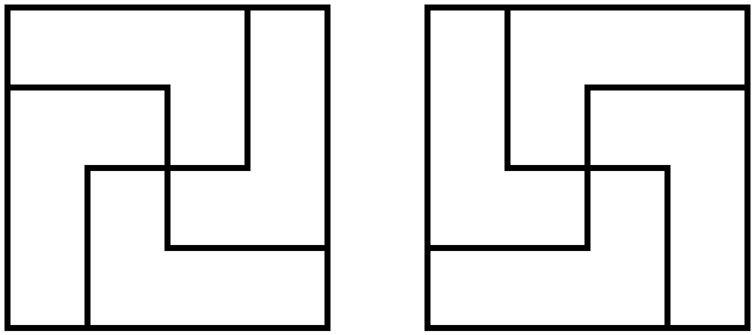
15. Tetris skills in action. We can use 2 L blocks to create a 2 x 4 rectangle, then two rectangles to create a 4 x 4 square. We can do this with vertical rectangles in 4 ways:



And with horizontal rectangles in 4 ways:



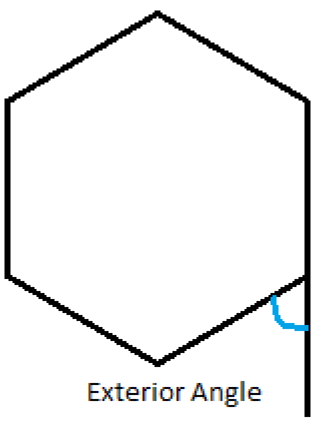
Finally, you can arrange the L blocks to create a square in 2 more ways:



So the answer is **E. 10**

16

The exterior angle of a polygon is the angle between a side and a line extended from a side next to it, like so:



All of the exterior angles in a simple shape add up to 360°. Also, all of the exterior angles of a regular polygon are equal to each other. So for an exterior angle of a regular polygon to be an exact degree measure, that measure must be a factor of 360. 1, 2, 3, 4, 5, and 6 all divide evenly into 360. 7 does not. So 7 is the smallest integer that cannot be the degree measure of the exterior angle of a regular polygon.

So the answer is **E. 7**

17

$\frac{m}{15} + \frac{n}{21} = \frac{7m+5n}{105}$

For $\frac{7m+5n}{105}$ to reduce, $7m + 5n$ must be a multiple of 3, 5, or 7 (the prime factors of 105). For $7m + 5n$ to be a multiple of 7, n would have to be a multiple of 7. But if n were a multiple of 7, then $\frac{n}{21}$ would not be in simplest terms. So $7m + 5n$ is not a multiple of 7. Likewise, $7m + 5n$ is not a multiple of 5, because then m would be a multiple of 5, and $\frac{m}{15}$ would not be in simplest terms. So we only need to look at the possibility of $7m + 5n$ being a multiple of 3.

$\frac{7m+5n}{3} = 2m + n + \frac{m+2n}{3}$, so $7m + 5n$ is a multiple of 3 if $2m + n$ is a multiple of 3. Because $\frac{m}{15}$ and $\frac{n}{21}$ are in simplest terms the possible values of m are {1, 2, 4, 7, 8, 11, 13, and 14} and the possible values of n are {1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, and 20}. For m, {1, 4, 7, and 13} are 1 more than a multiple of 3, and {2, 8, 11, and 14} are 2 more than a multiple of 3. For n, {1, 4, 10, 13, 16, and 19} are 1 above a multiple of 3, and {2, 5, 8, 11, 17, and 20} are 2 above a multiple of 3.

If m is 1 above a multiple of 3 (call it 3k+1), and n is 1 above a multiple of 3 (call it 3j+1), then $2m + n$ becomes $2(3k + 1) + (3j + 1) = 6k + 2 + 3j + 1 = 3(2k + j + 1)$, which is a multiple of 3.

If m is 1 above and n is 2 above, $2m + n = 2(3k + 1) + 3j + 2 = 6k + 3j + 4 = 3(2k + j + 1) + 1$, which is one above a multiple of 3.

If m is 2 above and n is 1 above, $2m + n = 2(3k + 2) + 3j + 1 = 6k + 3j + 5 = 3(2k + j + 1) + 2$, which is two above a multiple of 3.

If m is 2 above and n is 2 above, $2m + n = 2(3k + 2) + 3j + 2 = 6k + 3j + 6 = 3(2k + j + 2)$, which is a multiple of 3.

So $\frac{7m+5n}{105}$ reduces if m is {1, 4, 7, or 13} and n is {1, 4, 10, 13, 16, or 19}, or if m is {2, 8, 11, or 14} and n is {2, 5, 8, 11, 17, or 20}.

The number of such combinations is $4 * 6 + 4 * 6 = 48$

So the answer is **B. 48**

18

$rs + t = 14$

$r + st = 13$

$rs + st + r + t = 27$	Add above equations
$(s + 1)(r + t) = 27$	factor

Since r, s, and t are integers, $(s + 1)$ and $(r + t)$ are integers and therefore they must also be factors of 27. The factors of 27 are 1, 3, 9, 27. So $(s + 1)$ can be 1, 3, 9, or 27, meaning s can be 0, 2, 8, or 26.

If $s = 0$

$r(0) + t = 14$

$r + (0)t = 13$

$t = 14$

$r = 13$

This gives the solution (13, 0, 14)

If $s = 2$

$r(2) + t = 14$

$r + (2)t = 13$

$-4r - 2t = -28$	first equation times -2 to set up addition
$-3r = -15$	addition of above two lines
$r = 5$	divide by -3
$2(5) + t = 14$	substitution into first equation
$10 + t = 14$	simplify
$t = 4$	subtract 10

This gives the solution (5, 2, 4)

If $s = 8$

$8r + t = 14$

$r + 8t = 13$

$-64r - 8t = -112$

$-63r = -99$

$r = \frac{99}{63} = \frac{11}{7}$

So this solution is not in integers.

If $s = 26$

$26r + t = 14$

$r + 26t = 13$

$-676r - 26t = -364$

$-675r = -351$

$r = \frac{351}{675} = \frac{13}{25}$

So this solution is not in integers.

So there are 2 non-negative integer triplets which satisfy the system: (13, 0, 14), and (5, 2, 4).

So the answer is **A. 2**

19

If we define x^2 to be the middle term of our 17 perfect squares, then the x terms will drop out:

Using the formula for average:

$$\frac{(x-8)^2+(x-7)^2+\cdots(x-1)^2+x^2+(x+1)^2+\cdots(x+7)^2+(x+8)^2}{17}$$

Expanding:

$$\frac{x^2-16x+64+x^2+14x+49+\cdots x^2-2x+1+x^2+x^2+2x+1+\cdots x^2+14x+49+x^2+16x+64}{17}$$

Grouping like terms:

$$\frac{17x^2+16x-16x+14x-14x\ldots+2x-2x+64+64+49+49+\cdots 1+1}{17}$$

x terms cancel, grouping the squares:

$$\frac{17x^2+2(64+49+36+25+16+9+4+1)}{17}$$

Arithmetic:

$$\frac{17x^2+408}{17}$$

$x^2 + 24$

So $k = 24$

$2^r m = 24 = 8 * 3$

m is odd, so can't be a multiple of two, 2^r is a power of 2 so can't have factors other than 2. So 2^r is 8 and m is 3

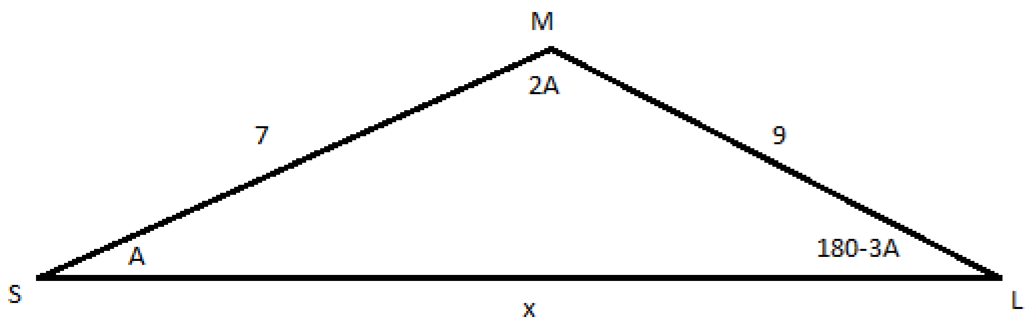
$2^r = 8$

$r = 3$

So the answer is **D. 3**

20.

Let us define angle S as A. Then angle M is equal to 2A (given in problem) and angle L is $180 - 3A$ (the angles of a triangle add to 180).



$\frac{7}{9} = \frac{\sin(180-3A)}{\sin(A)}$	Law of sines
$\frac{7}{9} = \frac{\sin 3A}{\sin A}$	$\sin(180 - x) = \sin x$
$\frac{7}{9} = \frac{\sin A \cos 2A + \sin 2A \cos A}{\sin A}$	expansion ($\sin 3A = \sin(A + 2A)$)
$\frac{7}{9} = \frac{\sin A \cos 2A + 2\sin A \cos^2 A}{\sin A}$	expand sin 2A
$\frac{7}{9} = \cos 2A + 2\cos^2 A$	reduce by sin A
$\frac{7}{9} = \cos^2 A - \sin^2 A + 2\cos^2 A$	expand cos 2A
$\frac{7}{9} = 3\cos^2 A - \sin^2 A$	simplify
$1 = \cos^2 A + \sin^2 A$	Pythagorean Identity
$\frac{16}{9} = 4\cos^2 A$	addition of above 2 equations
$\frac{4}{9} = \cos^2 A$	divide by 4 on both sides
$\frac{2}{3} = \cos A$	square root of both sides
$\frac{x}{9} = \frac{\sin 2A}{\sin A}$	Law of sines

$\frac{x}{9} = \frac{2 \sin A \cos A}{\sin A}$ expand sin 2A

$x = 18 \cos A$ reduce by sin A, multiply both sides by 9

$x = 18(\frac{2}{3})$ substitution

$x = 12$

So the answer is **C. 12**